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Calculation of Hydrodynamic Impulse Response Functions using Filon Quadrature

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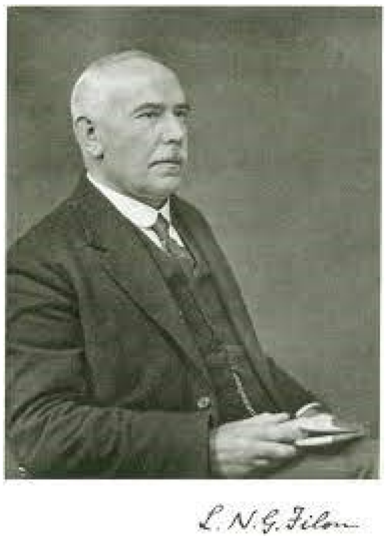

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SUMMARY

A method is shown for calculating hydrodynamic impulse response functions for ships and floating structures, using Filon quadrature.

Two alternative formulations are considered for the impulse response functions: one using cosine functions and one using sine functions. It is shown that the sine method is preferable, as it is more reliable at small time values.

The convergence of the Filon method at large time values is compared with Simpson's Rule. It is shown that the Filon method converges smoothly to zero, while Simpson's Rule becomes erratic and may not tend towards zero.

It is shown that Simpson's Rule can still be used to calculate impulse response functions, by using a very fine frequency spacing.

Overall, the Filon method is preferable for calculating impulse response functions, as it is robust and computationally efficient.

CONTENTS

1. Introduction

Frequency-domain ship motion theory uses linear “added mass” and “damping” coefficients, which are a function of oscillation frequency. For time-domain calculations, where it is impossible to specify a single oscillation frequency, “impulse response functions” are used instead. Impulse response functions have historically been used in many areas of physics, but appear to have been first applied to ship motions by Cummins (1962).

Time-dependent equations for ship motions are typically of the form (Cummins 1962 eq. 61, van Oortmerssen 1973 eq. 4.23, Gourlay 2019 eq. 1):

$$\sum_{j=1}^{6}\left[M_{ij}+A_{ij}(\infty)\right]\ddot{x}_j = X_i \;-\sum_{j=1}^{6} C_{ij}x_j - \int_0^{\infty}\sum_{j=1}^{6} L_{ij}(\tau)\ddot{x}_j(t-\tau)d\tau \tag{1}$$

The six motion degrees of freedom are:
x_1= “surge” (fore-aft motion at centre of gravity, positive forward)
x_2 = “sway” (transverse motion at centre of gravity, positive to port)
x_3 = “heave” (vertical motion at centre of gravity, positive upwards)
x_4 = “roll” (angle, positive to starboard)
x_5 = “pitch” (angle, positive bow-down)
x_6 = “yaw” (angle, positive bow-to-port).

Other symbols are defined as follows:
M_{ij} = mass matrix (Newman 1992, p. 307)
$A_{ij}(\infty)$ = added mass at infinite frequency
X_i = wave load in degree of freedom $i = 1, \dots, 6$
C_{ij} = hydrostatic restoring coefficients
$L_{ij}(\tau)$ = acceleration-based impulse response functions

The impulse response functions may be calculated from the added mass coefficients (A_{ij}) and damping coefficients (B_{ij}), which are functions of angular frequency (ω), by using either of the following relations (WAMIT 2019, equations 13.5, 13.6):

$$L_{ij}(t) = \frac{2}{\pi}\int_0^{\infty}\left[A_{ij}(\omega)-A_{ij}(\infty)\right]\cos(\omega t)\,d\omega \tag{2}$$

$$L_{ij}(t) = \frac{2}{\pi}\int_0^{\infty}\frac{B_{ij}(\omega)}{\omega}\sin(\omega t)\,d\omega \tag{3}$$

2. Filon quadrature

The integrands in equation (2) and (3) are highly oscillatory, when t is large. The classical quadrature technique of Simpson's Rule (see e.g. Press et al. 1999, p130) approximates the integrand by a parabola, over each double-interval. However, when t becomes large, the integrand oscillates so quickly that it cannot be reasonably approximated by a parabola.

A quadrature technique developed by Filon (1930), for sine and cosine integrals, approximates the multiplying function by a parabola over each double-interval, then does the resulting integration analytically. In this way, it does not matter if the whole integrand is highly oscillatory, as long as the multiplying function is slowly-varying.

A similar method (Tuck 1967) approximates the multiplying function as a trapezoid over each interval, then does the resulting integration analytically. Here we shall use the original (1930) method.

To apply Filon's method to impulse response functions, we start by writing Filon's quadrature formula (Filon 1930, eq.14) in terms of complex variables. This allows us to calculate cosine or sine integrals, by taking the real or imaginary part, respectively. Filon's equation is written:

$$\int_{\omega_0}^{\omega_{2N}} f(\omega)\ e^{i\omega t}\ d\omega = \sum_{n=0}^{2N} W_n\ f(\omega_n)\ e^{i\omega_n t}\ \Delta\omega \qquad (4)$$

The discrete frequencies are:

$$\omega_n = \omega_0 + n\,\Delta\omega \qquad (5)$$

The weighting functions are:

$$W_n = \begin{cases} \frac{\beta}{2} + i\alpha & n = 0 \\ \frac{\beta}{2} - i\alpha & n = 2N \\ \gamma & n = [1:2:2N-1] \\ \beta & n = [2:2:2N-2] \end{cases} \qquad (6)$$

The constants are:

$$\alpha = \frac{1}{\theta} + \frac{\sin(2\theta)}{2\theta^2} - \frac{2\sin^2(\theta)}{\theta^3} \qquad (7)$$

$$\beta = 2\left(\frac{1}{\theta^2} + \frac{\cos^2(\theta)}{\theta^2} - \frac{\sin(2\theta)}{\theta^3}\right) \qquad (8)$$

$$\gamma = 4\left(\frac{\sin(\theta)}{\theta^3} - \frac{\cos(\theta)}{\theta^2}\right) \qquad (9)$$

$$\theta = t\Delta\omega \qquad (10)$$

We can now evaluate equation (2), by taking the real part of equation (4), and setting:

$$f(\omega) = \frac{2}{\pi}\left[A_{ij}(\omega) - A_{ij}(\infty)\right] \qquad (11)$$

We can also evaluate equation (3), by taking the imaginary part of equation (4), and setting:

$$f(\omega) = \frac{2}{\pi}\frac{B_{ij}(\omega)}{\omega} \qquad (12)$$

3. Example calculation

We use the test case of a tanker moored at an open berth in shallow water. This case was tested at model scale by van Oortmerssen (1973) and analyzed using MoorMotions software by Gourlay (2019). Added mass and damping coefficients were first calculated at 51 equally-spaced oscillation frequencies of [0.00 : 0.05 : 2.50] rad/s, plus infinite frequency. Here we shall concentrate on the motion component of sway. Frequency-dependent added mass and damping coefficients are shown in Appendix A.

The sway impulse response function $L_{22}(t)$, as calculated using the cosine and sine formulations (equations 2 and 3) is shown in Figure 1.

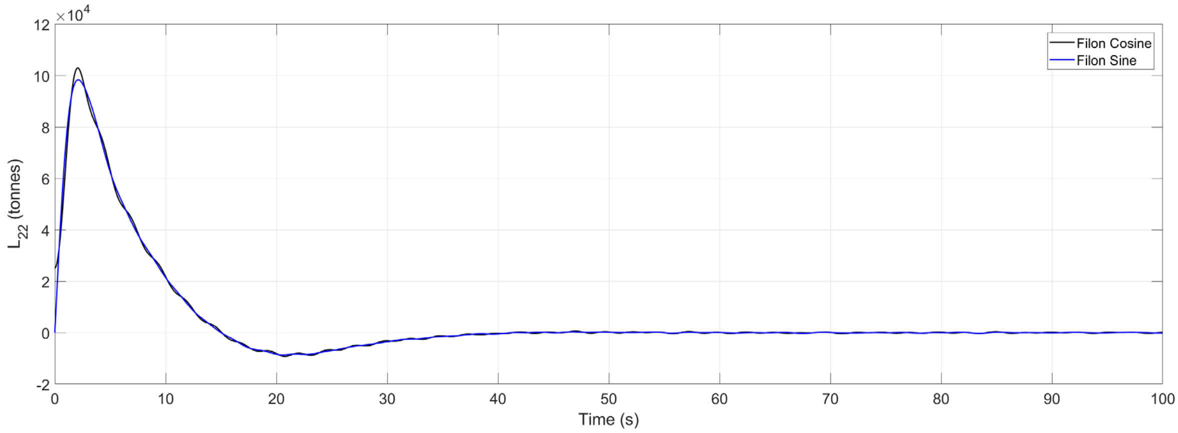

Figure 1: Sway impulse response function for van Oortmerssen test case, calculated using Filon quadrature with cosine and sine formulations

We expect that the impulse response functions should all tend toward zero as $t \to 0$, as can be seen analytically from equation (3). However, the cosine formulation fails to do this. It can be seen from equation (2) that, unless the added mass coefficients exactly average out to the infinite-frequency value, the impulse response function will not tend to zero as $t \to 0$. Due to the requirement to use a finite range of frequencies to calculate the added mass coefficients, it is to be expected that this formulation will not in general tend to zero as $t \to 0$. Therefore, the sine formulation is preferred, as it will always tend smoothly to zero as $t \to 0$.

The effectiveness of Filon quadrature can be seen by comparing to a standard quadrature (Simpson's Rule), as shown in Figure 2. We see that Simpson's Rule fails to converge at large time values, when the integrand in equation (3) becomes highly oscillatory.

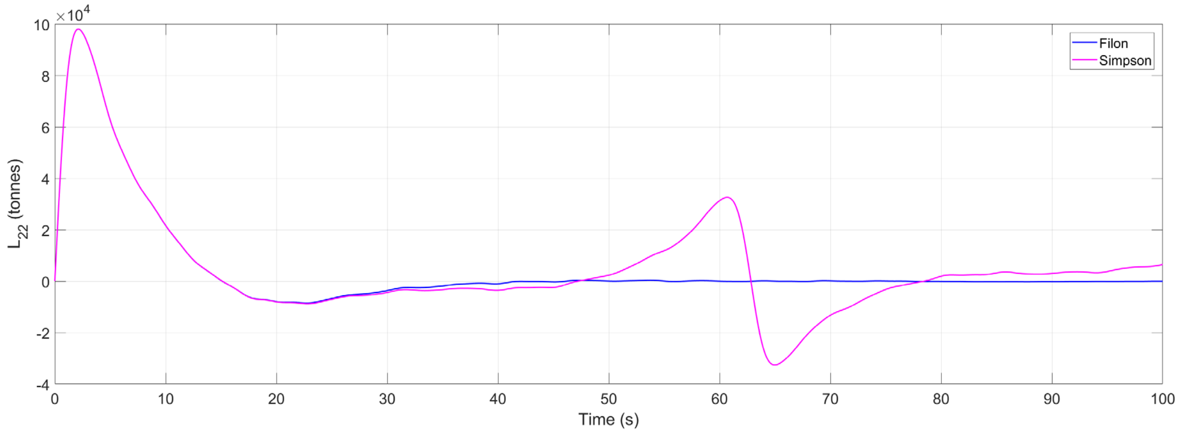

Figure 2: Comparison between Filon quadrature and Simpson's Rule

By using a sufficiently fine frequency interval, it is still possible to achieve good results using Simpson's Rule. For example, using a 5 times finer frequency spacing of 0.01 rad/s in our test case, allows the Simpson's Rule results in Figure 2 to converge correctly to zero, over the 100-second time period shown.

Finally, Figure 3 shows a comparison between Filon quadrature and WAMIT's f2t utility (WAMIT 2019, Chapter 13). Good agreement is found between both methods.

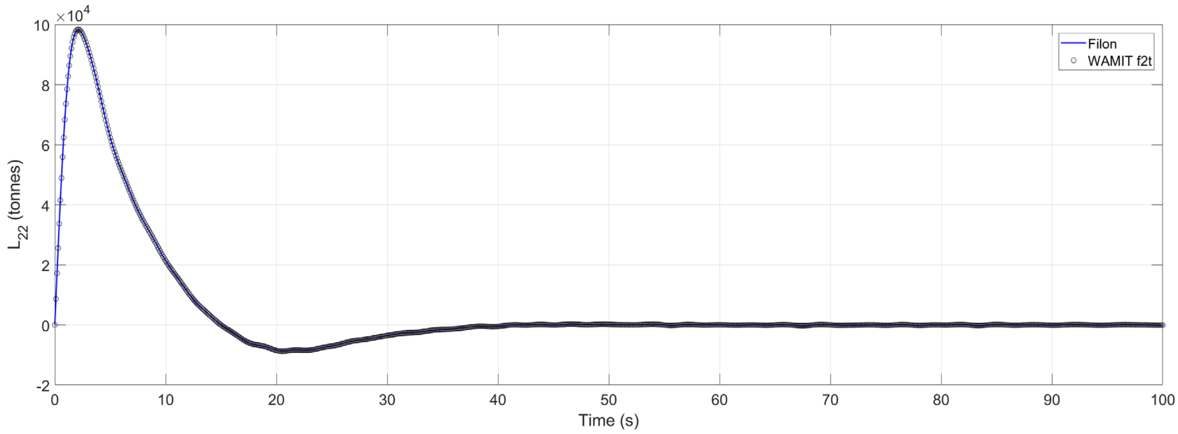

Figure 3: Comparison between Filon quadrature and WAMIT's f2t utility

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Appendix A – Added mass and damping coefficients

Sway added mass and damping coefficients, as used to develop the impulse response functions, are shown in Figure 4.

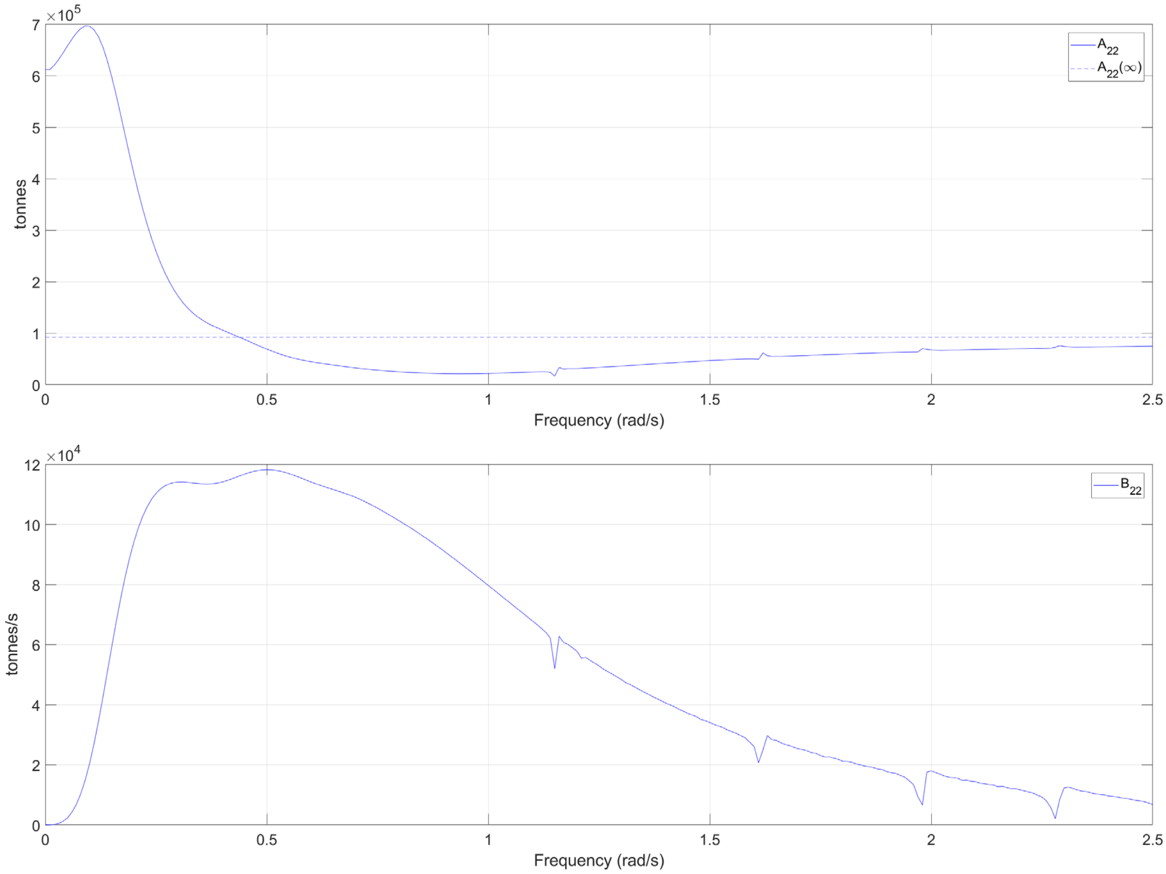

Figure 4: Sway added mass (top) and damping (bottom), as calculated using WAMIT for the van Oortmerssen test case. Spikes at irregular frequencies, a common feature of shallow-water calculations, are visible at higher frequencies.

The ratio of damping to frequency, as used in equation (3), is shown in Figure 5.

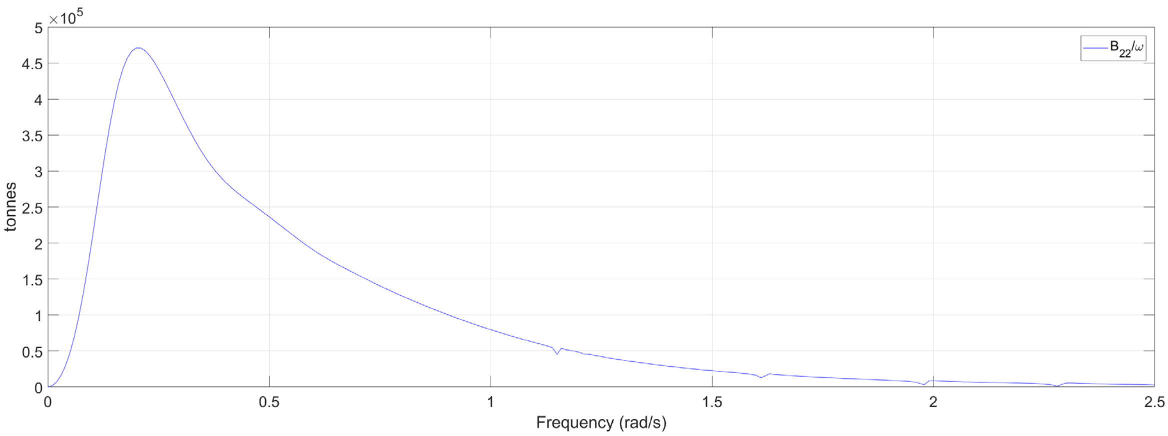

Figure 5: Ratio of damping to frequency, as used in equation (3)