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The Maximum Sinkage of a Ship

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A ship moving steadily forward in shallow water of constant depth h is usually subject to downward forces and hence squat, which is a potentially dangerous sinkage or increase in draft. Sinkage increases with ship speed, until it reaches a maximum at just below the critical speed $\sqrt{gh}$. Here we use both a linear transcritical shallow-water equation and a fully dispersive finite-depth theory to discuss the flow near that critical speed and to compute the maximum sinkage, trim angle, and stern displacement for some example hulls.

Introduction

For a thin vertical-sided obstruction extending from bottom to top of a shallow stream of depth h and infinite width, Michell (1898) showed that the small disturbance velocity potential $\Phi(x, y)$ satisfies the linearized equation of shallow-water theory (SWT)

$$\beta\Phi_{xx} + \Phi_{yy} = 0 \tag{1}$$

where $\beta = 1 - F_h^2$, with $F_h = U/\sqrt{gh}$ the Froude number based on x-wise stream velocity U and water depth h. This is the same equation that describes linearized aerodynamic flow past a thin airfoil (see e.g., Newman 1977 p. 375), with F_h replacing the Mach number. For a slender ship of a general cross-sectional shape, Tuck (1966) showed that equation (1) is to be solved subject to a body boundary condition of the form

$$\Phi_y(x, 0_\pm) = \pm\frac{US'(x)}{2h} \tag{2}$$

where $S(x)$ is the ship's submerged cross-section area at station x. The boundary condition (2) indicates that the ship behaves in the (x, y) horizontal plane as if it were a symmetric thin airfoil whose thickness $S(x)/h$ is obtained by averaging the ship's cross-section thickness over the water depth. There are also boundary conditions at infinity, essentially that the disturbance velocity $\nabla\Phi$ vanishes in subcritical flow ($\beta > 0$), or else behaves like an outgoing wave in supercritical flow ($\beta < 0$).

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As in aerodynamics, the solution of (1) is straightforward for either fully subcritical flow (where it is elliptic) or fully supercritical flow (where it is hyperbolic). In either case, the solution has a singularity as $\beta \to 0$, or $F_h \to 1$. In particular the subcritical (positive upward) force is given by Tuck (1966) as

$$F = \frac{\rho U^2}{2\pi h\sqrt{1-F_h^2}} \int\int dx\, d\xi\, B'(x)\, S'(\xi)\, \log|x-\xi| \tag{3}$$

with $B(x)$ the local beam at station x. Here and subsequently the integrations are over the wetted length of the ship, i.e., $-L/2 < x < L/2$ where L is the ship's waterline length.

This force F is usually negative, i.e., downward, and for a fore-aft symmetric ship, the resulting midship sinkage is given hydrostatically by

$$s = \left(\frac{V}{L^2}\right) C_S\, \frac{F_h^2}{\sqrt{1-F_h^2}} \tag{4}$$

where $V = \int S(x)dx$ is the ship's displaced volume, and

$$C_S = -\frac{L^2}{2\pi A_W V} \int\int dx\, d\xi\, B'(x) S'(\xi) \log|x-\xi| \tag{5}$$

where $A_W = \int B(x)dx$ is the ship's waterplane area. The nondimensional coefficient $C_S \approx 1.4$ has been shown by Tuck & Taylor (1970) to be almost a universal constant, depending only weakly on the ship's hull shape.

Hence the sinkage appears according to this linear dispersionless theory to tend to infinity as $F_h \to 1$. However, in practice, there are dispersive effects near $F_h = 1$ which limit the sinkage, and which cause it to reach a maximum value at just below the critical speed.

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Accurate full-scale experimental data for maximum sinkage are scarce. However, according to linear inviscid theory, the maximum sinkage is directly proportional to the ship length for a given shape of ship and depth-to-draft ratio (see later). This means that model experiments for maximum sinkage (e.g., Graff et al 1964) can be scaled proportionally to length to yield full-scale results, provided the depth-to-draft ratio remains the same.

The magnitude of this maximum sinkage is considerable. For example, the Taylor Series A3 model studied by Graff et al (1964) had a maximum sinkage of 0.89% of the ship length for the depth-to-draft ratio $h/T = 4.0$. This corresponds to a midship sinkage of 1.88 meters for a 200 meter ship. Experiments on maximum squat were also performed by Du & Millward (1991) using NPL round bilge series hulls. They obtained a maximum midship sinkage of 1.4% of the ship length for model 150B with $h/T = 2.3$. This corresponds to 2.8 meters midship sinkage for a 200 meter ship. Taking into account the fact that there is usually a significant bow-up trim angle at the speed where the maximum sinkage occurs, the downward displacement of the stern can be even greater, of the order of 4 meters or more for a 200-meter long ship.

It is important to note that only ships that are capable of traveling at transcritical Froude numbers will ever reach this maximum sinkage. Therefore, maximum sinkage predictions will be less relevant for slower ships such as tankers or bulk carriers. Since the ships or catamarans that frequently travel at transcritical Froude numbers are usually comparatively slender, we expect that slender-body theory will provide good results for the maximum sinkage of these ships.

For ships traveling in channels, the width of the channel becomes increasingly important around $F_h = 1$, when the flow is unsteady and solitons are emitted forward of the ship (see e.g., Wu & Wu 1982). Hence experiments performed in channels cannot be used to accurately predict maximum sinkage for ships in open water. The experiments of Graff et al were done in a wide tank, approximately 36 times the model beam, and are the best results available with which to compare an open-water theory. However, even with this large tank width, sidewalls still affect the flow near $F_h = 1$, as we shall discuss.

Transcritical shallow-water theory (TSWT)

It is not possible simply to set $\beta = 0$ in (1) in order to gain useful information about the flow near $F_h = 1$. As with transonic aerodynamics, it is necessary to include other terms that have been neglected in the linearized derivation of SWT (1).

An approach suggested by Mei (1976) (see also Mei & Choi, 1987) is to derive an evolution equation of Korteweg-de Vries (KdV) type for the flow near $F_h = 1$. The usual one-dimensional forms of such equations contain both nonlinear and dispersive terms. It is not difficult to incorporate the second space dimension y into the derivation, resulting in a two-dimensional KdV equation, which generalizes (1) by adding two terms to give

$$\beta\Phi_{xx} + \Phi_{yy} - \frac{3}{U}\Phi_x\Phi_{xx} + \frac{1}{3}h^2\Phi_{xxxx} = 0 \qquad (6)$$

The nonlinear term in $\Phi_x\Phi_{xx}$ but not the dispersive term in Φ_{xxxx} was included by Lea & Feldman (1972). Further solutions of this nonlinear but nondispersive equation were obtained by Ang (1993) for a ship in a channel. Chen & Sharma (1995) considered the unsteady problem of soliton generation by a ship in a channel, using the Kadomtsev-Petviashvili equation, which is essentially an unsteady version of equation (6). Although they concentrated on finite-width domains, their method is also applicable to open water, albeit computationally intensive. Further nonlinear and dispersive terms were included by Chen (1999), allowing finite-width results to be computed over a larger range of Froude numbers.

Mei (1976) considered the full equation (6) in open water and provided an analytic solution for the linear case where the term $\Phi_x\Phi_{xx}$ is omitted. He showed that for sufficiently slender ships the nonlinear term in equation (6) is of less importance than the dispersive term and can be neglected; also that the reverse is true for full-form ships where the nonlinear term is dominant. This is also discussed in Gourlay (2000).

As stated earlier, most ships that are capable of traveling at transcritical speeds are comparatively slender. For these ships it is dispersion, not nonlinearity, that limits the sinkage in open water. Nonlinearity is usually included in one-dimensional KdV equations by necessity, as a steepening agent to provide a balance to the broadening effect of the dispersive term in Φ_{xxxx}. In open water, however, there is already an adequate balance with the two-dimensional term in Φ_{yy}. This is in contrast to finite-width domains, which tend to amplify transcritical effects and cause the flow to be more nearly unidirectional. Hence nonlinearity becomes important in finite-width channels to such an extent that steady flow becomes impossible in a narrow range of speeds close to critical (see e.g., Constantine 1961, Wu & Wu 1982).

Therefore, for slender ships in shallow water of large or infinite width, we can solve for maximum squat using the simple transcritical shallow-water (TSWT) equation

$$\beta\Phi_{xx} + \Phi_{yy} + \gamma\Phi_{xxxx} = 0 \qquad (7)$$

(writing $\gamma = h^2/3$), subject to the same boundary condition (2). The term in γ provides dispersion that was absent in the SWT, and limits the maximum sinkage.

Solution

The solution of the TSWT equation (7) for the potential Φ, subject to the boundary condition (2), can be written (Mei 1976) as a Fourier integral

$$\Phi(x, y) = \frac{Ui}{4\pi h}\int d\xi S(\xi)\int_{-\infty}^{\infty}\frac{k}{\lambda}e^{ik(\xi-x)-\lambda|y|}dk \qquad (8)$$

where $\lambda^2 = \beta k^2 - \gamma k^4$. If $\beta > 0$ and $k^2 < \beta/\gamma$, the parameter λ can be taken as real positive, and the interpretation of (8) is straightforward. However, if $\beta < 0$, or if $\beta > 0$ and $k^2 > \beta/\gamma$, then λ^2 is negative and we must choose the correct sign of its pure-imaginary square root. The basis for this choice is that the solution must then represent outgoing not incoming waves, and this is so if we take

$$\lambda = \begin{cases} -i\sqrt{\gamma k^4 - \beta k^2}, & \beta k^2 - \gamma k^4 < 0 \text{ and } k > 0 \\ +i\sqrt{\gamma k^4 - \beta k^2}, & \beta k^2 - \gamma k^4 < 0 \text{ and } k < 0 \end{cases} \qquad (9)$$

With that interpretation, the k-integrand in (8) is defined for all β and all k, excepting the integrable singularities at $k = \pm\sqrt{\beta/\gamma}$.

Pressure, forces, and squat

The pressure excess over hydrostatic is given by the linearized Bernoulli equation as $p = -\rho U \Phi_x$. Matching with the inner flow as in Tuck (1966) gives the leading order pressure on the hull as

$$p(x) = -\rho U \Phi_x|_{y=0}$$

or

$$p(x) = -\frac{\rho U^2}{4\pi h} \int d\xi S(\xi) \lim_{y\to 0} \int_{-\infty}^{\infty} \frac{k^2}{\lambda} e^{ik(\xi-x)-\lambda|y|}\, dk \tag{10}$$

The (positive upward) vertical force on the ship is $F = \int p(x) \times B(x) dx$; thus

$$F = -\frac{\rho U^2}{4\pi h} \int_{-\infty}^{\infty} \frac{k^2}{\lambda} \overline{S}(k) \overline{B}^*(k) dk \tag{11}$$

Similarly the trim moment $M = -\int x p(x) B(x) dx$ (measured positive bow-up) becomes

$$M = \frac{\rho U^2}{4\pi h} \int_{-\infty}^{\infty} \frac{k^2}{\lambda} \overline{S}(k) \overline{xB}^*(k) dk \tag{12}$$

where

$$\begin{aligned} \overline{S}(k) &= \int S(x) e^{ikx} dx \\ \overline{B}(k) &= \int B(x) e^{ikx} dx \\ \overline{xB}(k) &= \int xB(x) e^{ikx} dx \end{aligned} \tag{13}$$

are Fourier transforms of $S(x)$, $B(x)$ and $xB(x)$ respectively, a star denoting complex conjugate. Again, these integrals are taken over the wetted length of the ship.

Mei (1976) expressed the transcritical force in a form similar to (3), where the logarithm is replaced by a kernel involving Bessel and Struve functions. However, for computational purposes, the Fourier-transformed version (11) appears to be preferable.

We note using the identity

$$\begin{aligned} &\frac{1}{2} \int_{-\infty}^{\infty} |k| \overline{S}(k) \overline{B}^*(k)\, dk \\ &= -\int \int dx\, d\xi\, B'(x) S'(\xi) \log|x-\xi| \end{aligned} \tag{14}$$

that, away from $F_h = 1$, the TSWT vertical force (11) agrees with the SWT result (3) in the limit as $\gamma \to 0$, i.e., when transcritical dispersion effects are negligible. In performing this limit, we must scale γ using the ship's length L, so that we are in essence taking the limit as $h/L \to 0$. Similarly the trim moment (12), and therefore actual sinkage and trim, all agree with SWT in the limit $h/L \to 0$.

Once the vertical force and trim moment are known, we can solve simultaneously for the (downward) midship sinkage s and bow-up trim angle θ using the hydrostatic equilibrium relations

$$\begin{aligned} -\frac{F}{\rho g} &= \int \{s + x\theta\} B(x)\, dx \\ \frac{M}{\rho g} &= \int x\{s + x\theta\} B(x)\, dx \end{aligned} \tag{15}$$

Symmetric hulls

To illustrate some properties of the TSWT, we shall now consider the special case of ships with fore-aft symmetry. In that case, $\overline{B}$ and $\overline{S}$ are real-valued even functions of k, and the star in (11) is unnecessary. The integrand in (11) is odd in k for supercritical flow ($\beta < 0$), so that the sinkage force is identically zero. For subcritical flow, the contributions from $k > \sqrt{\beta/\gamma}$ and $k < -\sqrt{\beta/\gamma}$, where λ is imaginary, similarly cancel each other, so (11) reduces to

$$F = -\frac{\rho U^2}{2\pi h} \int_0^{\sqrt{\beta/\gamma}} \frac{k}{\sqrt{\beta - \gamma k^2}} \overline{S}(k) \overline{B}(k) dk \tag{16}$$

Thus the transcritical sinkage force is a predominantly subcritical-like or elliptic phenomenon, depending on the long-wave part $|k| < \sqrt{\beta/\gamma}$ of the wave number range where all disturbances due to the ship tend to zero at infinity. This is consistent with the result of Tuck (1966) that the dispersionless supercritical sinkage vanishes for ships with fore-aft symmetry.

Conversely, the subcritical trim moment for fore-aft symmetric vessels can be written

$$M = \frac{\rho U^2}{2\pi h} \int_{\sqrt{\beta/\gamma}}^{\infty} \frac{ik}{\sqrt{\gamma k^2 - \beta}} \overline{S}(k) \overline{xB}(k) dk \tag{17}$$

since $\overline{xB}(k)$ is pure imaginary and odd in k. There is zero contribution to the moment from the range $|k| < \sqrt{\beta/\gamma}$. (For supercritical flow the lower terminal $\sqrt{\beta/\gamma}$ in (17) is replaced by zero.)

Hence transcritical trim is a predominantly supercritical-like or hyperbolic phenomenon, depending on large wave numbers where the ship produces an outgoing short wave at infinity. Again, this is consistent with the trim vanishing in subcritical dispersionless flow for fore-aft symmetric ships (Tuck 1966), and taking a large bow-up value in supercritical flow, which is anticipated in the transcritical range.

For fore-aft symmetry, the hydrostatic coupling between sinkage and trim vanishes, and (15) becomes

$$\begin{aligned} F &= -\rho g A_W s \\ M &= \rho g I_W \theta \end{aligned} \tag{18}$$

where $A_W = \int B(x)\, dx$ and $I_W = \int x^2 B(x)\, dx$. Therefore the sinkage and trim displacements are given by

$$\begin{aligned} s &= \frac{F_h^2}{2\pi A_W} \int_0^{\sqrt{\beta/\gamma}} \frac{k}{\sqrt{\beta - \gamma k^2}} \overline{S}(k)\, \overline{B}(k)\, dk \\ \theta &= \frac{F_h^2}{2\pi I_W} \int_{\sqrt{\beta/\gamma}}^{\infty} \frac{ik}{\sqrt{\gamma k^2 - \beta}} \overline{S}(k)\, \overline{xB}(k)\, dk \end{aligned} \tag{19}$$

Finite-width channel

In the case of a ship moving along the center of a channel of width $2w$, assuming steady flow we can still solve (7) using Fourier-transform techniques. This time we use a wall boundary condition on $y = \pm w$ to obtain the following expressions for the vertical force and trim moment:

$$F = -\frac{\rho U^2}{4\pi h} \int_{-\infty}^{\infty} \frac{k^2}{\lambda} \coth(\lambda w)\, \overline{S}(k)\, \overline{B}^*(k)\, dk$$
$$M = \frac{\rho U^2}{4\pi h} \int_{-\infty}^{\infty} \frac{k^2}{\lambda} \coth(\lambda w)\, \overline{S}(k)\, \overline{xB}^*(k)\, dk \qquad (20)$$

Here λ has the same definition as for the open water TSWT.

In addition to the singularities at $k = \pm\sqrt{\beta/\gamma}$, for $|k| > \sqrt{\beta/\gamma}$ there are singularities in $\coth(\lambda w)$ at

$$k^2 = \frac{\beta}{2\gamma}\left(1 + \sqrt{1 + \frac{4n^2\pi^2\gamma}{w^2\beta^2}}\right) \qquad (21)$$

for integer values of n. All of these singularities, including $k = \pm\sqrt{\beta/\gamma}$, are simple poles. The path of integration must pass above these poles in the complex k-plane in order that there be no disturbance as $x \to -\infty$.

We can see that the numerical evaluation of (20) is complicated. Although the sum of the residues is straightforward to evaluate (and in fact this sum is zero for symmetric hulls), the Cauchy principal-value part of the integral is difficult, due to the infinite number of unequally spaced poles. Even for symmetric hulls there is no cancelation for $|k| > \sqrt{\beta/\gamma}$; the integrand is even in k, whereas in infinite width it is odd.

In any case, we expect that a steady finite-width theory of this sort will be of limited validity. Linearized slender-body theory is unable to predict the onset of unsteadiness, which is an important feature of transcritical channel flow. Only when F_h is not too close to 1, or in very wide channels, is the flow still steady and this theory applicable.

Finite-depth theory

The difficulties inherent in solving the flow field for $F_h \approx 1$ can also be overcome by removing the shallowness assumption completely. This method was introduced by Tuck & Taylor (1970), who used slender-body theory to describe the flow around a ship moving in a general (not necessarily shallow) depth of water. Here we make some important improvements on their original method and compute sinkage and trim for ships with or without fore-aft symmetry.

Tuck & Taylor found that the vertical force and trim moment are each the sum of two components, i.e.,

$$F = F_\infty + F_d$$
$$M = M_\infty + M_d \qquad (22)$$

Here F_∞ and M_∞ are the force and moment on the lower half of an equivalent double body in an unbounded fluid; F_d and M_d are a correction due to the finite depth of water.

The calculation of F_∞ and M_∞ is a well known but difficult problem. Here we approximate F_∞ using the analytic formula of Havelock (1939) for a spheroid, approximating each hull as an equivalent slender spheroid of the same length and displacement, as done by Tuck & Taylor (1970). This gives for F_∞

$$F_\infty = \rho U^2 A_W \epsilon^2 \left(\log\frac{\epsilon}{2} + \frac{3}{2} - \epsilon\right) \qquad (23)$$

with A_W again the waterplane area. The slenderness ϵ is equal to the beam/length ratio of the equivalent spheroid; for a general hull this is given by

$$\epsilon = \sqrt{\frac{12}{\pi} C_\forall} \qquad (24)$$

Here $C_\forall$ is the volumetric coefficient V/L^3, with V the displaced volume.

For fore-aft symmetric ships, M_∞ is identically zero. Since the hull shapes that we are considering are close to fore-aft symmetric, we shall not attempt to calculate M_∞, assuming instead that it is small enough to be neglected. This becomes valid for general ships in the shallow-water limit, as the depth correction M_d has been shown (Tuck & Taylor 1970) to formally dominate M_∞ in shallow water.

The depth corrections F_d and M_d satisfy

$$F_d = -\frac{\rho U^2}{4\pi^2} \int_{-\infty}^{\infty} k^2\, \overline{S}(k)\, \overline{B}^*(k)\, A(k)\, dk$$
$$M_d = \frac{\rho U^2}{4\pi^2} \int_{-\infty}^{\infty} k^2\, \overline{S}(k)\, \overline{xB}^*(k)\, A(k)\, dk \qquad (25)$$

where $\overline{S}$, $\overline{B}$ and $\overline{xB}$ are as in (13). Here $A(k)$ is given by

$$A(k) = -2\int_{|k|}^{\infty} \frac{1}{\sqrt{q^2 - k^2}} \times \left[1 + \frac{q}{F_h^2 k^2 h - q\tanh(qh)}\right] dq \qquad (26)$$

The integrand in (26) has a simple pole at $q = q_0$, where

$$q_0 \tanh(q_0 h) = F_h^2 k^2 h \qquad (27)$$

In order that there be no waves far upstream, this pole must be avoided by passing above it in the complex q-plane when $k > 0$, and beneath it when $k < 0$ (Gourlay 2000).

Since the integrand in (26) is pure real, the real part of $A(k)$ can be written as a Cauchy principal value integral, and the imaginary part can be evaluated from the residue at the pole. This gives

$$\operatorname{Im}\{A(k)\} = \frac{-2\pi q_0 \operatorname{sgn}(k)}{\sqrt{q_0^2 - k^2}\left(q_0 h \operatorname{sech}^2(q_0 h) + \tanh(q_0 h)\right)} \qquad (28)$$

which corrects the Tuck & Taylor (1970) result.

We note that $\operatorname{Re}\{A(k)\}$ is even and $\operatorname{Im}\{A(k)\}$ is odd (as they must be for F to be pure real in (25)). Hence we need only discuss (26) for $k > 0$.

The pole in (26) will only occur in the range of integration if $q_0 > k$. For $F_h > 1$ this is always true, and the imaginary part of $A(k)$ as given by (28) is nonzero but finite for all k. The real part can be evaluated by a simple numerical integration once a substitution is made to remove the singularity at $q = k$.

When $F_h < 1$, q_0 is only greater than k for $k > k_0$, where

$$\frac{\tanh(k_0 h)}{k_0 h} = F_h^2 \qquad (29)$$

This imaginary part of $A(k)$ is zero for $k < k_0$ and given by (28) for $k > k_0$. Since $q_0 = k$ when $k = k_0$, (28) is singular around

$k = k_0$. Similarly, the Cauchy principal-value integral in (26) is unbounded when q_0 and k coincide, as the denominator acts like $(q-k)^{-3/2}$ around $q = k$. The full (q, k) double integrals in (25) do still exist, but we must take care in evaluating them around $(q, k) = (q_0, k_0)$. The imaginary part of $A(k)$ for $F_h < 1$ was not included by Tuck & Taylor (1970), leading to the incorrect conclusion of zero subcritical trim for fore-aft symmetric hulls.

As stated by Tuck & Taylor, the corrections F_d and M_d are inversely proportional to depth in the shallow-water limit, and therefore formally dominate F_∞ and M_∞ as $h/L \to 0$. It can also be shown by taking the $h/L \to 0$ limit of equation (26) that, away from $F_h = 1$, the vertical force and moment agree with the simple shallow-water theory (SWT). In the transcritical region $F_h \approx 1$, Gourlay (2000) has shown that the finite-depth theory agrees with TSWT in the shallow water limit.

Scaling

The TSWT force (11) and moment (12) were derived in terms of dimensional variables. However, they may be written in dimensionless form, since they and the TSWT and FDT theories described are all linear. For a given shape of ship, the dimensionless products

$$\frac{F}{\rho g S_{\max} B_{\max}} \quad \text{and} \quad \frac{M}{\rho g S_{\max} B_{\max} L} \tag{30}$$

are functions only of F_h and h/L. Solving (15) then tells us that

$$\frac{s}{L} = C_\forall \, f\left(F_h, \frac{h}{L}\right)$$
$$\theta = C_\forall \, m\left(F_h, \frac{h}{L}\right) \tag{31}$$

where the functions f, m depend only on the shape of ship.

Therefore, for a given shape of ship, h/L ratio and Froude number, the ship may be stretched in the streamwise or transverse directions, with the only influence on s/L coming from $C_\forall$. In particular, s/L is independent of scale, if the ship's proportions are kept constant. This allows us to justifiably "scale up" model test results according to this theory. Since the trim angle θ is independent of scale, the bow and stern sinkages are also proportional to ship length for constant values of h/L and F_h.

In FDT, only the force and moment corrections F_d and M_d (25) can be written so as to allow stretching in the transverse directions. F_∞ and M_∞ both have complicated dependence upon beam and section area and so cannot be scaled in this way. However, the total FDT force and moment can still be written as dimensionless products (30), provided the proportions of the ship remain constant. Therefore s/L and θ are still independent of scale in FDT, and can be justifiably scaled up with increasing ship size.

Results

Comparing TSWT and FDT

In order to compare TSWT and FDT numerically, we first consider a fore-aft symmetric hull with parabolic waterplane and section area curves, i.e.,

$$B(x) = B_{\max}\left(1 - \frac{4x^2}{L^2}\right)$$
$$S(x) = S_{\max}\left(1 - \frac{4x^2}{L^2}\right) \tag{32}$$

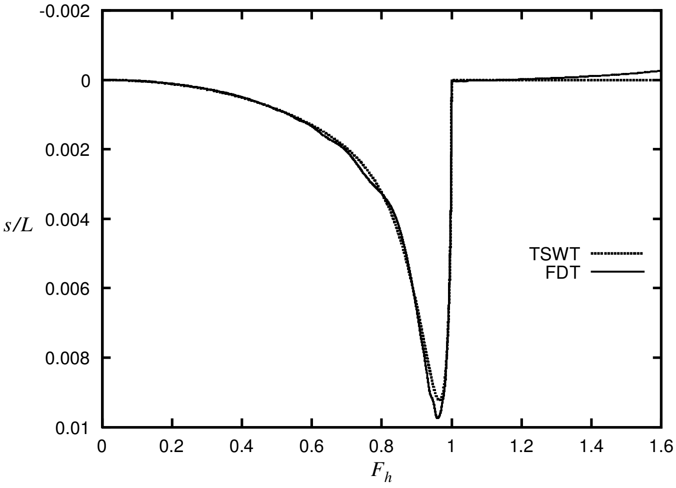

Fig. 1 Sinkage (as a fraction of ship length) as a function of F_h for parabolic hull, using TSWT and FDT

The maximum beam and section area are chosen to be in the same proportions as the Taylor A3 hull, which will be discussed later. Therefore we have chosen $L/B_{\max} = 10.8$ and $L^2/S_{\max} = 378.3$ for these results.

Figure 1 shows the scaled midship sinkage s/L as a function of F_h for this hull, with $h/L = 0.125$, as calculated using both transcritical theories. We see that, at this relatively small value of h/L, the two theories produce very similar results. The sinkage in each case rises to a maximum at $F_h = 0.965$ before tending sharply back toward zero. For TSWT it can be shown analytically that the sinkage tends to zero in a square-root manner as $F_h \to 1$, and that it is identically zero for $F_h > 1$ as in SWT (Tuck 1966). FDT has similar properties, although there is an apparent small but non-zero supercritical sinkage which we believe to be spurious (see later).

We notice that the maximum sinkage according to FDT ($s/L = 0.0097$) is slightly higher than according to TSWT ($s/L = 0.0092$).

Whereas TSWT gives a smooth monotone increase in the sinkage until it reaches the maximum, FDT is slightly oscillatory in slope. This is a property of the governing equations and not of the numerical method, as the graph was reproduced exactly with different grid spacings in each of the variables. Being fully dispersive, FDT retains some bow-stern wave interference effects that are familiar in classical ship hydrodynamics, and which, for example, lead to analogous oscillations in wave resistance curves for ships.

The present FDT results can be compared with those of Tuck & Taylor (1970). The maximum sinkage given there for one particular value of h/L agrees moderately well (within 10%) with our predictions, although their sinkage curve is more oscillatory at

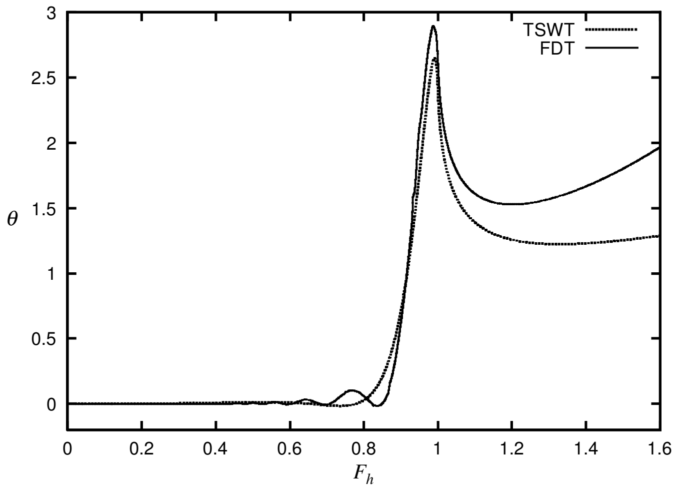

Fig. 2 Bow-up trim angle (in deg) as a function of F_h for parabolic hull with $h)L = 0.125$, using TSWT and FDT

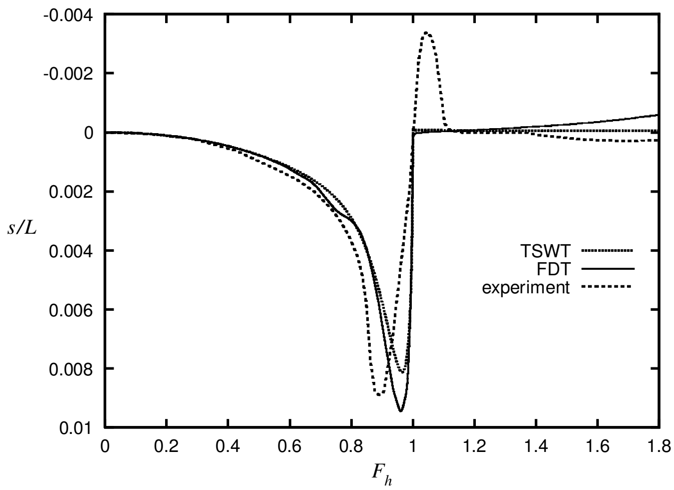

Fig. 3 Sinkage (as a fraction of ship length) as a function of F_h for Taylor Series hull, with $h)L = 0.125$. TSWT, FDT, and experimental results

lower Froude numbers. This may have been due to the computations in Tuck & Taylor (1970) not handling the singularity in $A(k)$ for $F_h < 1$ with adequate numerical precision.

A problem with FDT is an anomalous behavior at large Froude numbers. The infinite-depth force contribution F_∞ increases in proportion to U^2; if the theory is to agree with TSWT in the shallow water limit, F_∞ should be canceled by the correction F_d at high Froude numbers, leading to a net force that tends to zero at infinite speed. However, any error in approximating either F_∞ or F_d will ultimately produce a total force that is erroneously quadratic in U for large F_h. This is already beginning to be apparent at the right side of Fig. 1, and the present FDT results (especially with use of the Havelock approximation to F_∞) are not to be considered accurate at large F_h.

Figure 2 shows the corresponding trim angles for the same hull, according to the two theories. According to both theories, the ship will have a large bow-up trim angle in the neighborhood of $F_h = 1$. The maximum trim occurs at $F_h = 0.99$ according to both theories, which is slightly higher than the Froude number at which the maximum sinkage occurs ($F_h = 0.965$).

As is the case with sinkage, the maximum trim according to FDT (2.88 deg) is slightly higher than according to TSWT (2.65 deg). Also, the TSWT curve is smooth, while the FDT is oscillatory at lower Froude numbers. The discrepancy for large Froude numbers cannot be due to an error in approximating M_∞, since this is identically zero. We suspect that the correction M_d may be quite sensitive for large Froude numbers, and place more faith in the TSWT in this range.

Comparison with experimental results

We have used both the TSWT and FDT to compute the midship sinkage of a "Taylor standard series" hull (Gertler 1954), which corresponds to a slightly non-fore-aft-symmetric ship. A model hull of that shape designated A3 by Graff et al (1964) was used; this has $L/B = 10.8$, $L/T = 32.4$ (with T the draft) and volumetric coefficient $C_\text{V} = 0.0017$. The experimental results of Graff et al (1964) for the same hull are reproduced here for comparison.

Figure 3 shows the midship sinkage as a function of Froude number for the A3 hull, traveling in water of scaled depth $h/L = 0.125$. As an example, this would correspond to a 200 meter ship with a draft of 6.2 meters, traveling in water of depth 25 meters. The sinkage is calculated using TSWT and FDT theories, with the experimental results also shown.

We see that both theories agree reasonably well with the experimental results. In particular, the maximum sinkage determined experimentally ($s/L = 0.0089$) lies in between that predicted by TSWT ($s/L = 0.0081$) and by FDT ($s/L = 0.0094$). There is a difference in the Froude number at which this maximum sinkage occurs, being at $F_h = 0.89$ experimentally, although we predict it at $F_h = 0.965$, according to both theories.

The rising of the ship in the water, for F_h slightly greater than one, is not predicted by either of the theories. For larger F_h, the predicted sinkage is very close to zero, in accordance with the experimental results. Since this ship is not quite fore-aft symmetric, TSWT gives a very small but nonzero sinkage for $F_h > 1$, which agrees reasonably with the experimental results. The quadratic-type error in FDT again becomes apparent for large F_h.

At this point we must question whether the experimental results of Graff et al (1964), which were obtained in a tank of large but finite width, adequately approximate open water. The numerical results of Chen & Sharma (1995) and Chen (1999), both based on unsteady theory, predict that unsteady solitons of significant amplitude would be generated in such a tank width in a range of Froude numbers roughly between 1 and 1.07. These unsteady oscillations, which were actually witnessed in the experimental results of Graff et al (1964) at low values of h/L, prevented them from giving any results for $h/L < 0.125$.

According to the numerical results of Chen & Sharma (1995), the effect of such solitons on sinkage and trim is considerable in that narrow range of Froude numbers. Specifically, it seems likely that the negative supercritical sinkage observed in the results of Graff et al (1964) may be solely a finite-width effect.

According to a recent study by Sharma & Chen (2000) which concentrated on the effect of tank width, the Froude number at

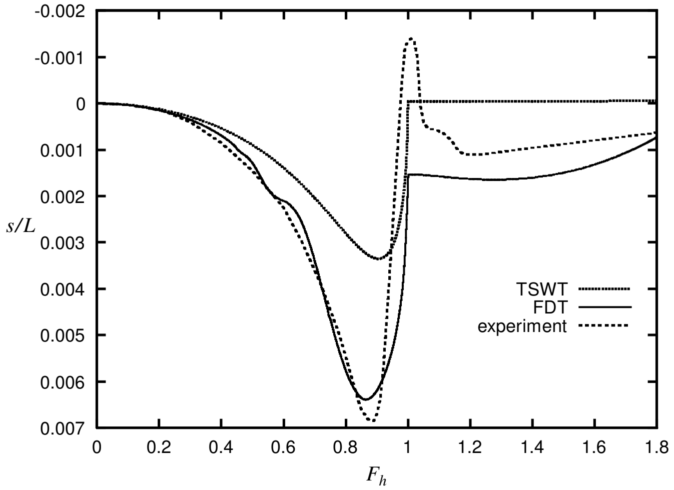

Fig. 4 Sinkage (as a fraction of ship length) as a function of F_h for Taylor Series hull, with $h)L = 0.25$. TSWT, FDT, and experimental results

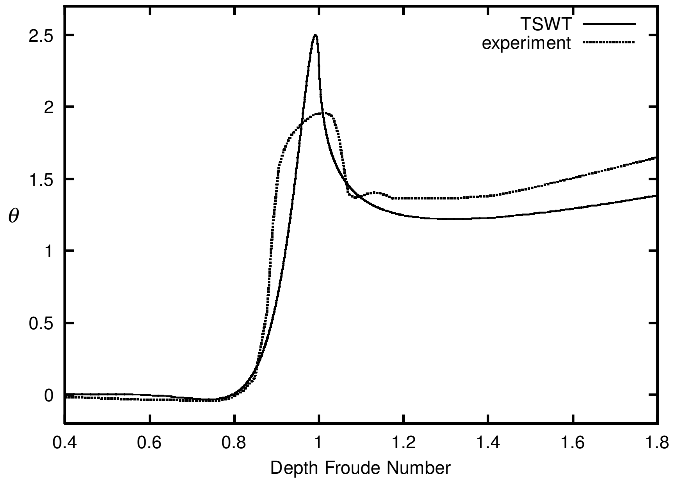

Fig. 5 Bow-up trim angle (in deg) as a function of F_h for Taylor Series hull, with $h)L = 0.125$. Comparison of TSWT with experiment

which the maximum sinkage occurs is seen to decrease as the tank width decreases. Therefore the discrepancy in the Froude number at which the maximum sinkage occurs may also be due to finite-width effects in the experimental results. The present predictions of the Froude number for maximum sinkage may in fact be more accurate as estimations for fully-open water than any observations from finite-width tank experiments.

Figure 4 shows the midship sinkage for the same ship traveling in twice the depth of water, i.e., $h/L = 0.25$. This corresponds to a depth of 50 meters for our example hull of length 200 meters and draft 6.2 meters. In this case we see that TSWT significantly underestimates the maximum sinkage, indicating that the shallow-water assumption that h/L is small is no longer acceptable. FDT, however, still gives a maximum sinkage of $s/L = 0.0066$ at $F_h = 0.86$, which is impressively accurate compared to the experimental maximum of $s/L = 0.0068$ at $F_h = 0.88$. Clearly the FDT is appropriate and accurate for such non-shallow-water depths.

Trim and stern sinkage

The trim of the A3 hull, as predicted using TSWT, was also compared to experimental results. This is shown in Fig. 5. There is fair agreement in this case, despite the predicted peak being sharper and higher than the experimental results. It is thought that this discrepancy may also be due to finite-width effects in the experimental results, as numerical results of Sharma & Chen (2000) show a broader, flatter peak for this width of channel compared to wider channels.

The large bow-up trim experienced by ships in the transcritical speed range means that the ship's stern is particularly vulnerable to grounding for Froude numbers very close to 1. Since the maximum midship sinkage and maximum trim occur at close to the same Froude number, the stern sinkage will be very large around this Froude number.

In Fig. 6 we have plotted the scaled stern sinkage s_{stern}/L (given by $s/L + \theta/2$, with θ in radians) for the A3 and parabolic hulls. This is found using TSWT and plotted as a function of Froude number. We see that for the A3 hull the maximum stern sinkage is $s_{\text{stern}}/L = 0.0285$. The experimental results give a smaller maximum of $s_{\text{stern}}/L = 0.0228$, the difference mainly being due to the difference in trim angles.

For the parabolic hull, which has a larger C_∇, the stern sinkage reaches $s_{\text{stern}}/L = 0.0308$. For a 200 meter ship, these results correspond to a predicted stern sinkage of 5.7 meters for the A3 hull (compared to 4.56 meters according to the finite-width experimental results) and 6.16 meters for the parabolic hull.

Maximum sinkage

We have seen that when midship and stern sinkage are plotted against F_h for a fixed value of h/L, they both reach a maximum just below $F_h = 1$. In fact, at very high supercritical Froude numbers, the stern sinkage may reach a higher value than in the transcritical range; however, the speeds required for this to occur are not realistic for most displacement hulls.

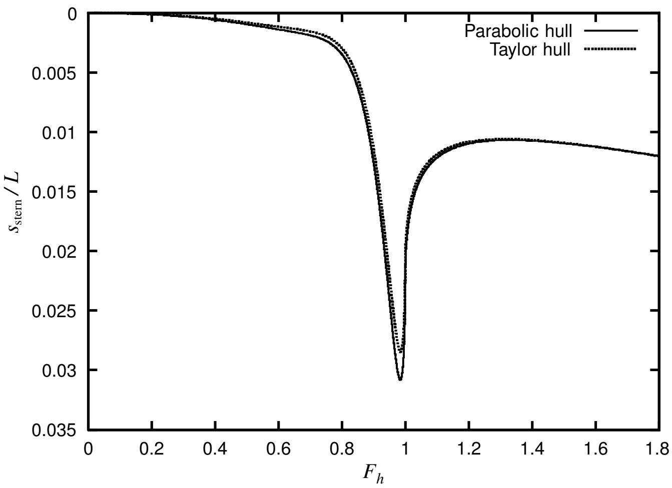

Fig. 6 Stern sinkage (as a fraction of ship length) as a function of F_h for parabolic and Taylor Series hull, with $h)L = 0.125$. Calculated using TSWT

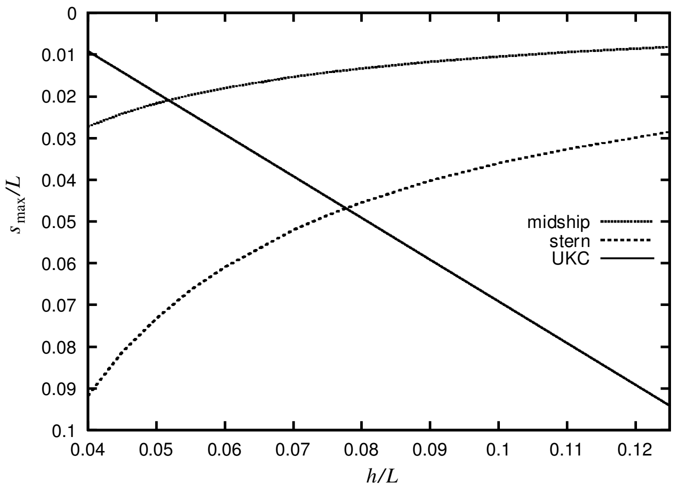

Fig. 7 Maximum midship and stern sinkage (as a fraction of ship length) as a function of $h)L$ for Taylor Series hull, as calculated using TSWT. At-rest underkeel clearance (UKC) also shown for each depth

By finding the maximum sinkage with respect to F_h for each value of h/L, we can plot the maximum midship and stern sinkage as a function of h/L. This is shown in Fig. 7 for the Taylor A3 hull. The at-rest underkeel clearance (UKC) is also shown for each depth.

Where the maximum sinkage is larger than the UKC, there exists a range of Froude numbers for which the ship is predicted to scrape the sea floor. Therefore, the point in Fig. 7 where the UKC line crosses the maximum stern sinkage curve gives the value of h/L for which the stern is predicted to just touch the sea floor at one particular Froude number. On the other hand, if the ship were unable to trim, the midship would touch the sea floor at the value of h/L for which the maximum midship sinkage curve crosses the UKC line.

Therefore we see that for the Taylor A3 hull the stern is at risk of grounding for $h/L < 0.078$, which corresponds to $h/T < 2.52$. Similarly, if trim were not taken into account, the midship would be at risk of grounding for $h/L < 0.052$, which corresponds to $h/T < 1.68$. These are large at-rest clearances, but are necessary for ships which intend to travel at transcritical speeds.

Conclusions

We have used two slender-body methods to solve for the sinkage and trim of a ship traveling at arbitrary Froude number, including the transcritical region.

The transcritical shallow water theory (TSWT) developed by Mei (1976) has been extended and exploited numerically, using numerical Fourier transform methods to give sinkage and trim via a double numerical integration. This theory has also been extended to the case of a ship moving in a channel of finite width; however, the numerical difficulty in computing the resulting force integral, and its limited validity, mean that the open-water theory is more practically useful.

The finite-depth theory (FDT) developed by Tuck & Taylor (1970) has also been improved and used for general hull shapes. This theory gives a sinkage force and trim moment that are slightly oscillatory in F_h. Since the theory involves summing infinite-depth and finite-depth contributions, both of which vary with U^2 at high Froude numbers, any error will grow approximately quadratically with U. Therefore we cannot use this theory at large supercritical Froude numbers. Also, the difficulty in finding the infinite-depth contributions numerically, as well as the extra numerical integration needed to compute the force and moment, make the FDT slightly more difficult to implement than TSWT.

When comparing our open water theoretical results to the experimental results of Graff et al (1964) in a wide channel, both theories were seen to give good results in the shallow water case $h/L = 0.125$. The main discrepancies between the theoretical and experimental results at this depth were: the trim predicted using TSWT has a sharper, higher peak than was found experimentally; the maximum sinkage was predicted to occur at a slightly higher Froude number than was observed experimentally; and neither of the theories predicted the rise of the ship in the water that was observed experimentally at low supercritical Froude numbers.

All of these discrepancies are explained qualitatively by the effect of the channel walls on the experimental results, which means that the theory looks very promising for predicting transcritical squat in open water. However, we cannot properly judge the accuracy of the method without true open water results with which to compare.

One departure from the experimental results that is not explained by finite-width effects is the lesser angle of trim predicted by TSWT at supercritical Froude numbers. Although sinkage is basically an inviscid phenomenon, it is thought that viscosity may affect the trim; boundary-layer separation near the stern would tend to decrease the pressure there and increase the bow-up trim moment slightly.

In the deeper-water case $h/L = 0.25$, for which the channel walls have less effect on the experimental results, we found that the shallow-water TSWT was outside its range of validity. Meanwhile, however, the fully-dispersive FDT accurately predicts the maximum sinkage and the Froude number at which this occurs.

In practice, scenarios in which ships are at risk of grounding will normally have $h/L < 0.125$. Since the TSWT is a shallow-water theory and it works well at $h/L = 0.125$, we expect that it will give even better results at smaller, practically useful values of h/L. Also, since the TSWT and FDT give almost identical results for $h/L < 0.125$, and the TSWT is a much simpler theory, we recommend it as a simple and accurate method for predicting transcritical squat in open water. Of particular interest to mariners is the maximum stern sinkage, which, as we here demonstrated, can be remarkably and dangerously large at close to critical speed.

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References

Ang, W. T. 1993 Nonlinear sinkage and trim for a slender ship in shallow water of finite width. (Internal report, University of Adelaide).

Chen, X.-N. 1999 Hydrodynamics of wave-making in shallow water. Ph.D. thesis, University of Stuttgart.

Chen, X.-N. and Sharma, S. D. 1995 A slender ship moving at a near-critical speed in a shallow channel, *Journal of Fluid Mechanics*, **291**, 263.

Constantine, T. 1961 On the movement of ships in restricted waterways. *Journal of Fluid Mechanics*, **9**, 247.

Du, Y.-Z. and Millward, A. 1991 The effect of squat on the wave resistance of a fast round bilge hull in shallow water. *International Shipbuilding Progress*, **38**, 416, 341.

Gertler, M. 1954 A reanalysis of the original test data for the Taylor standard series. David Taylor Model Basin Report, **806**, Department of the Navy, Washington, DC.

Gourlay, T. P. 2000 Mathematical and computational techniques for predicting the squat of ships. Ph.D. thesis, University of Adelaide.

Graff, W., Kracht, A., and Weinblum, G. 1964 Some extensions of D. W. Taylor's standard series. *Transactions*, SNAME, **72**, 374.

Havelock, T. H. 1939 Note on the sinkage of a ship at low speeds. *Z. Angew. Math. Mech.*, **19**, 202.

Lea, G. K. and Feldman, J. P. 1972 Transcritical flow past slender ships. *Proceedings*, 9th Symposium on Naval Hydrodynamics, ONR, Washington, DC, 1527.

Mei, C. C. 1976 Flow around a thin body moving in shallow water. *Journal of Fluid Mechanics*, **77**, 737.

Mei, C. C. and Choi, H. S. 1987 Forces on a slender ship advancing near the critical speed in a wide canal. *Journal of Fluid Mechanics*, **179**, 59.

Michell J. H. 1898 The wave resistance of a ship. *Philosophical Magazine*, **45**, 106.

Newman, J. N. 1977 *Marine Hydrodynamics*, MIT Press, Cambridge, Mass.

Sharma, S. D. and Chen, X.-N. 2000 Interaction of ship waves with river banks and uneven bottoms. *Proceedings*, 4th International Conference on Hydrodynamics, Yokohama, 157.

Tuck, E. O. 1966 Shallow water flows past slender bodies. *Journal of Fluid Mechanics*, **26**, 81.

Tuck, E. O. and Taylor, P. J. 1970 Shallow water problems in ship hydrodynamics. *Proceedings*, 8th Symposium on Naval Hydrodynamics, ONR, Washington, DC, 627.

Wu, D. M. and Wu, T. Y. 1982 Three-dimensional nonlinear long waves due to moving surface pressure. *Proceedings*, 14th Symposium on Naval Hydrodynamics, ONR, Washington, DC, 103.